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Secant Tree Calculus

Dominique Foata and Guo-Niu Han

Abstract. A true Tree Calculus is being developed to make a joint study of the two statistics "eoc" (end of minimal chain) and "pom" (parent of maximum leaf) on the set of secant trees. Their joint distribution restricted to the set $\{\mathrm{eoc} - \mathrm{pom} \le 1\}$ is shown to satisfy two partial difference equation systems, to be symmetric and to be expressed in the form of an explicit three-variable generating function.

1. Introduction

As was done in our previous paper, whose purpose was to evaluate the distribution of a two-variable statistic defined on the set of *tangent trees* [FH13], we pursue the same goal with the secant trees, by using an appropriate *Tree Calculus*. The secant trees, defined in full detail in the sequel, are labeled binary trees; they are called *secant*, because the number of them with $2n$ nodes is equal to the *secant* number E_{2n}, namely, the coefficient of $u^{2n}/(2n)!$ in the Taylor expansion of $\sec u$:

$$\begin{aligned}(1.1)\quad \sec u = \frac{1}{\cos u} &= \sum_{n\ge 0} \frac{u^{2n}}{(2n)!} E_{2n}\\ &= 1 + \frac{u^2}{2!}1 + \frac{u^4}{4!}5 + \frac{u^6}{6!}61 + \frac{u^8}{8!}1385 + \frac{u^{10}}{10!}50521 + \cdots\end{aligned}$$

(see, e.g., [Ni23, p. 177-178], [Co74, p. 258-259]). The purpose of this paper is to calculate the *joint distribution* of the two statistics 'eoc" (end of minimal chain) and "pom" (parent of maximum leaf) on the set of secant trees. To achieve this and in particular derive an explicit three-variable generating function two *partial difference equation systems* are to be solved. The solutions of those systems are based on *Tree Calculus*, which consists of partitioning each set of secant trees into smaller subsets, and developing a natural algebra on those subsets for solving the difference equations.

1.1. *The secant trees.* The trees in question form a subclass of the so-called *increasing, binary* trees. The latter trees having n nodes labeled $1, 2, \ldots, n$ are in one-to-one correspondence with the $n!$ ordinary permutations of the sequence $12\cdots n$. See, for instance, Viennot [Vi88, chap. 3].

Key words and phrases. Tree Calculus, partial difference equations, increasing trees, complete increasing trees, secant and tangent trees, end of minimal chain, parent of maximum leaf, bivariate distributions, secant numbers.

Mathematics Subject Classifications. 05A15, 05A30, 11B68.

Their actual definition is next stated by using the traditional vocabulary on trees, such as node, leaf, child, root, … In particular, when a node is not a leaf, it is said to be an *interior node*.

Definition. For each positive integer n an *increasing* tree of size n is defined by the following axioms:

(1) it is a *labeled* tree with n nodes, labeled $1, 2, \ldots, n$; the node labeled 1 being the *root*;

(2) each node has no child (then called a *leaf*), or one child, or two children;

(3) the label of each node is smaller than the label of its children, if any;

(4) the tree is planar and each child of each node is, either on the left (it is then called the *left child*), or on the right (the *right child*); moreover, the tree can be embedded on the Euclidean plane as follows: the root has coordinates $(0,0)$, the left child (if any) $(-1,1)$, the right child (if any) $(1,1)$, the grandchildren (if any) $(-3/2,2)$, $(-1/2,2)$, $(1/2,2)$, $(3/2,2)$, the greatgrandchildren (if any) $(-7/4,3)$, $(-5/4,3)$, … , $(7/4,3)$, etc. With this convention all the nodes have different abscissas. The node having the maximum abscissa is then defined in a unique manner. Call it the *rightmost node*; it is either a leaf, or a node having a left child, but no right child.

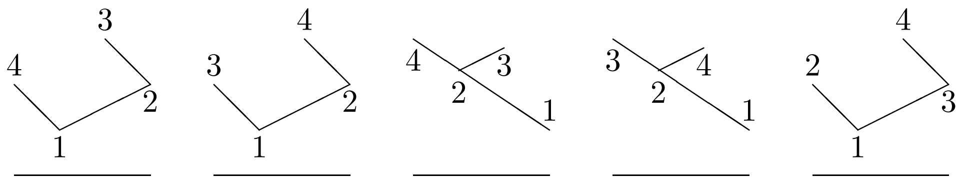
Consider the orthogonal projections of those n nodes onto a horizontal axis. Reading the labels of those projected n nodes from left to right yields a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ of $12\cdots n$. This *projection* defines a bijection of the set of all increasing trees with n nodes onto the permutation group $\mathfrak{S}_n$.

The trees which correspond to permutations $x_1x_2\cdots x_n$ having the property that $x_1 > x_2$, $x_2 < x_3$, $x_3 > x_4$, … , in an alternating way, usually named *alternating*, are called *complete increasing.* In an equivalent manner, an increasing tree with n node is said to be *complete* (see [Vi88], [KPP94], [St99]), if axioms (1)–(4) hold with the further property

(5) every node is, either a leaf, or a node with two children, except the rightmost node, which has one *left* child, but no right child when n is *even.* This rightmost node will then be referred to as being the *one-child node*.

Each complete increasing tree with n nodes is simply called *secant* (resp. *tangent*) whenever n is *even* (resp. *odd*). For each $n \geq 1$ let $\mathfrak{T}_n$ be the set of all complete increasing trees of size n. In Fig. 1.1 the five complete increasing trees from $\mathfrak{T}_4$, accordingly the five *secant* trees, have been drawn, together with the projections of their node labels on the horizontal axis. Notice that each of those projections, when read from left to right, forms an *alternating permutation* of the sequence $1\,2\,3\,4$. Under

each tree have been calculated the two statistics "eoc" and "pom" defined below.



	4 1 3 2	3 1 4 2	4 2 3 1	3 2 4 1	2 1 4 3
eoc =	3	4	3	2	3
pom =	1	2	2	3	2

Fig. 1.1

1.2. *The main two statistics.* Let $t \in \mathfrak{T}_n$ $(n \geq 1)$. If a node labeled a has two children labeled b and c, define $\min a := \min\{b, c\}$; if it has one child b, let $\min a := b$. The *minimal chain* of t is defined to be the sequence $a_1 \to a_2 \to a_3 \to \cdots \to a_{j-1} \to a_j$, with the following properties: (i) $a_1 = 1$ is the label of the root; (ii) for $i = 1, 2, \ldots, j-2$ the $(i+1)$-st term a_{i+1} is the label of an interior node and $a_{i+1} = \min a_i$; (iii) a_j is the node of a leaf. Define the "**e**nd **o**f the minimal **c**hain" of t to be $\mathrm{eoc}(t) := a_j$. If the leaf with the maximum label n is incident to a node labeled k, define its "**p**arent **o**f the **m**aximum leaf" to be $\mathrm{pom}(t) := k$. See Fig. 1.1.

Those two statistics have been introduced by Poupard [Po88] on *tangent* trees (n even). She proved that "eoc" and "1+pom" were equidistributed on each set $\mathfrak{T}_{2n+1}$ of tangent trees, also that their common univariable distribution satisfied a finite difference equation system; she further calculated their generating function. In [FH13] it was proved that the equidistribution actually holds on every set $\mathfrak{T}_n$ $(n \geq 1)$ by constructing an explict bijection ϕ of $\mathfrak{T}_n$ onto itself with the property that: $1+\mathrm{pom}(t) = \mathrm{eoc}\,\phi(t)$ for all t.

Working with alternating permutations it was shown in [FH12] that the distribution of each statistic "eoc," "pom" on *secant* trees satisfied the *same* finite difference system, as introduced by Poupard for tangent trees, but the initial conditions were different. Their generating function could also be calculated.

The next step was to study the *joint* distribution of the pair $(\mathrm{eoc}, \mathrm{pom})$ on each set $\mathfrak{T}_n$, that is to say, letting

$$f_n(m,k) := \#\{t \in \mathfrak{T}_n : \mathrm{eoc}(t) = m \text{ and } \mathrm{pom}(t) = k\} \tag{1.2}$$

see whether those numbers are solutions of a *partial* finite difference equation system, and try to calculate their generating function. This program has been achieved in our paper [FH13] for the sets of *tangent* trees $\mathfrak{T}_{2n+1}$. The purpose of this paper is then to pursue this program for *secant* trees.

1.3. *The joint distribution.* The first values of the matrices $M_{2n} := (f_{2n}(m,k))$ $(2\le m\le 2n; 1\le k\le 2n-1)$ are listed in Table 1.1. For each matrix have been evaluated the *row sums* $f_{2n}(m,\bullet) := \sum_k f_{2n}(m,k) = \#\{t\in \mathfrak{T}_{2n} : \mathrm{eoc}(t)=m\}$ (resp. *column sums* $f_{2n}(\bullet,k) := \sum_m f_{2n}(m,k) = \#\{t\in\mathfrak{T}_{2n} : \mathrm{pom}(t)=k\}$) on the rightmost column (resp. the bottom row). In the South-East-corner is written the total sum $f_{2n}(\bullet,\bullet) := \sum_{m,k} f_{2n}(m,k)$, equal to the *secant number* E_{2n}. Keeping in mind that alternating permutations are equidistributed with complete increasing trees, the identity

$$f_{2n}(\bullet,\bullet)=\sum_{m,k} f_{2n}(m,k)=E_{2n} \tag{1.3}$$

is a consequence of the old result derived by Désiré André [An1979, An1881]. Also, as "1+pom" and "eoc" are equidistributed, we have:

$$f_{2n}(\bullet,k-1)=f_{2n}(k,\bullet)\quad (2\le k\le 2n). \tag{1.4}$$

On each entry $f_{2n}(m,k)$ may be defined two *partial differences* with respect to m and k as follows:

$$\mathop{\Delta}_{m} f_{2n}(m,k) := f_{2n}(m+1,k)-f_{2n}(m,k); \tag{1.5}$$

$$\mathop{\Delta}_{k} f_{2n}(m,k) := f_{2n}(m,k+1)-f_{2n}(m,k). \tag{1.6}$$

By convention $f_{2n}(m,k):=0$ if $(m,k)\notin[2,2n]\times[1,2n-1]$. Our main results are the following.

Theorem 1.1. *The finite difference equation systems hold:*

$(R\,1)$ $\mathop{\Delta}_{m}^2 f_{2n}(m,k)+4\,f_{2n-2}(m,k-2)=0\ (2\le m\le k-3<k\le 2n-1);$

$(R\,2)$ $\mathop{\Delta}_{k}^2 f_{2n}(m,k)+4\,f_{2n-2}(m,k)=0\ (2\le m\le k-1<k\le 2n-3).$

The first two top rows and rightmost two columns of the upper triangles $\{f_{2n}(m,k) : 2\le m<k\le 2n-1\}$ $(n\ge 2)$ can be evaluated in function of the row or column sums $f_{2n-2}(\bullet,k)$, $f_{2n-2}(m,\bullet)$, as is now stated.

Theorem 1.2. *We have:*

(1.7) $\quad f_{2n}(2,k)=f_{2n-2}(\bullet,k-2)=f_{2n-2}(k-1,\bullet)\quad (3\le k\le 2n-1);$ *[First top row]*

$f_{2n}(3,k)=3\,f_{2n}(2,k)\quad (4\le k\le 2n-1);$ *[Second top row]*

$f_{2n}(m,2n-1)=f_{2n-2}(m,\bullet)=f_{2n-2}(\bullet,m-1)\quad (2\le m\le 2n-2);$ *[Rightmost column]*

$f_{2n}(m,2n-2)=3\,f_{2n}(m,2n-1)\quad (2\le m\le 2n-3).$ *[Next to rightmost column]*

$M_2 =$

$k=$	1	$f_2(m,.)$
$m=2$	1	1
$f_2(.,k)$	1	$E_2=1$

$M_4 =$

$k=$	1	2	3	$f_4(m,.)$
$m=2$	.	.	1	1
3	1	2	.	3
4	.	1	.	1
$f_4(.,k)$	1	3	1	$E_4=5$

$M_6 =$

$k=$	1	2	3	4	5	$f_6(m,.)$
$m=2$	.	.	1	3	1	5
3	1	2	.	9	3	15
4	3	7	10	.	1	21
5	1	4	8	2	.	15
6	.	2	2	1	.	5
$f_6(.,k)$	5	15	21	15	5	$E_6=61$

$M_8 =$

$k=$	1	2	3	4	5	6	7	$f_8(m,.)$
$m=2$	.	.	5	15	21	15	5	61
3	5	10	.	45	63	45	15	183
4	15	35	50	.	101	63	21	285
5	21	54	86	106	.	45	15	327
6	15	46	82	87	50	.	5	285
7	5	22	46	60	40	10	.	183
8	.	16	16	14	10	5	.	61
$f_8(.,k)$	61	183	285	327	285	183	61	$E_8=1385$

$M_{10} =$

$k=$	1	2	3	4	5	6	7	8	9	$f_{10}(m,.)$
$m=2$	.	.	61	183	285	327	285	183	61	1385
3	61	122	.	549	855	981	855	549	183	4155
4	183	427	610	.	1405	1575	1341	855	285	6681
5	285	720	1132	1466	.	1989	1575	981	327	8475
6	327	884	1460	1863	2050	.	1405	855	285	9129
7	285	836	1448	1838	1870	1466	.	549	183	8475
8	183	606	1110	1466	1490	1155	610	.	61	6681
9	61	288	588	854	950	804	488	122	.	4155
10	.	272	272	256	224	178	122	61	.	1385
$f_{10}(.,k)$	1385	4155	6681	8475	9129	8475	6681	4155	1385	$E_{10}=50521$

Table 1.1: the matrices M_{2n} $(1 \le n \le 5)$

Proposition 1.3. *We further have:*

$$f_2(\bullet,1)=1;\quad f_{2n}(\bullet,1)=f_{2n-2}(\bullet,\bullet)=E_{2n-2}\ (n\ge 2); \tag{1.8}$$

$$f_{2n}(\bullet,2)=3f_{2n-2}(\bullet,\bullet)=3E_{2n-2}\ (n\ge 2); \tag{1.9}$$

$$\mathop{\Delta^2}_{m} f_{2n}(m,\bullet)+4\,f_{2n-2}(m,\bullet)=0\quad (2\le m\le 2n-2); \tag{$R\,3$}$$

$$\mathop{\Delta^2}_{k} f_{2n}(\bullet,k)+4\,f_{2n-2}(\bullet,k)=0\quad (1\le k\le 2n-3). \tag{$R\,4$}$$

Theorem 1.4. *The previous two theorems and Proposition* 1.3 *provide an explicit algorithm for calculating the entries of the upper triangles of the matrices* $M_{2n}=(f_{2n}(m,k))$ $(n\ge 1)$.

The entries of the *lower triangles* $\{f_{2n}(m,k) : 1\le k<m\le 2n\}$ in Tables 1.1 have been calculated directly by means of formula (1.2). Contrary to the *upper triangles* we do not have any explicit numerical algorithm to get them; only the entries situated on the three sides of those lower triangles can be directly evaluated, as shown in Section 6. See, in particular, Table 6.1. For the *upper triangles*, we can derive an explicit generating function, as stated in the next theorem.

Theorem 1.5. *The triple exponential generating function for the upper triangles of the matrices* $(f_{2n}(m,k))$ *is given by*

$$\begin{aligned}\sum_{2\le m<k\le 2n-1} f_{2n}(m,k)\frac{x^{2n-k-1}}{(2n-k-1)!}\frac{y^{k-m-1}}{(k-m-1)!}\frac{z^{m-2}}{(m-2)!}\\ =\frac{\cos(2y)+2\,\cos(2(x-z))-\cos(2(z+x))}{2\,\cos^3(x+y+z)}.\end{aligned} \tag{1.10}$$

The right-hand side of (1.10) is symmetric in x,z. Hence, the change $x\leftrightarrow z$ in the left-hand side of (1.10) shows that

$$f_{2n}(2n+1-k,2n+1-m)=f_{2n}(m,k). \tag{1.11}$$

The upper triangles already mentioned are then symmetric with respect to their counter-diagonals. This result can also be extended as follows.

Theorem 1.6. *Let* $\mathrm{Up}(2n)$ *be the set of all* (m,k) *from* $[2,2n]\times[1,2n-1]$ *such that either* $m-1\le k$, $(m,k)=(3,1)$, *or* $(m,k)=(2n,2n-2)$. *Then* (1.11) *holds for every* $(m,k)\in\mathrm{Up}(2n)$.

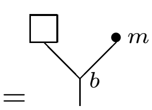
Those theorems and proposition are proved in the next sections, once the main ingredients on Tree Calculus have been developed, as done in the next section.

2. Tree Calculus

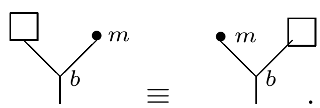
We adopt the following notations and conventions: for each triple (n, m, k) let $\mathfrak{T}_{2n,m,k}$ (resp. $\mathfrak{T}_{2n,m,\bullet}$, resp. $\mathfrak{T}_{2n,\bullet,k}$) denote the subset of $\mathfrak{T}_{2n}$ of all trees t such that $\mathrm{eoc}(t) = m$ and $\mathrm{pom}(t) = k$ (resp. $\mathrm{eoc}(t) = m$, resp. $\mathrm{pom}(t) = k$). Also, *symbols representing families of trees will also designate their cardinalities.* With this convention $\mathfrak{T}_{n,m,k} := \#\mathfrak{T}_{n,m,k}$. The matrix of the upper triangles $\mathfrak{T}_{2n,m,k}$ $(2 \le m < k \le 2n-1)$ will be denoted by $\mathrm{Upper}(\mathfrak{T}_{2n})$.

Subtrees (possibly empty) are indicated by the symbols "◯," "▽", or "□." The notation "⧅" (resp. "⦸," resp. "▽⃥") is used to indicate that the subtree "□" (resp. "◯," resp. "▽") *contains the one-child node or is empty.* Letters occurring below or next to subtrees are labels of their roots. The end of the minimal chain in each tree is represented by a bullet "•."

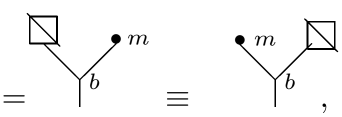
In the sequel certain families of secant trees will be represented by symbols, called *trunks*. For example, the symbol

$A =$ 

is the trunk that designates the *family* of all trees t from the underlying set $\mathfrak{T}_{2n}$ having a node b, parent of both a subtree "□" and the leaf m, which is also the end of the minimal chain. Notice that unlike the secant trees which are ordered, the trunk is *unordered*, so that

(2.1) 

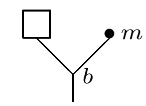
When the subtree "□" contains the one-child node or is empty, we let

$A(⧅) :=$ 

be the family of all trees t from the underlying set $\mathfrak{T}_{2n}$ having a node b, parent of the *right* child "⧅" and the *left* child leaf m. Let $A(\star)$ be the set of all trees from A such that "□" is not empty and does not contain the one-child node. We have the following decomposition

$$A = A(\star) + A(⧅). \tag{2.2}$$

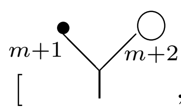
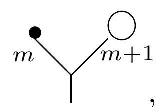
When a further condition (C) is imposed on a trunk A we shall use the notation $[A, (C)]$. For example, the symbol

$B = [$  $, a\,]$

is the trunk that has the same characteristic of the trunk A as above with the further property that the node labeled a belongs, *neither* to the subtree of root b, *nor* to the path going from root 1 to b.

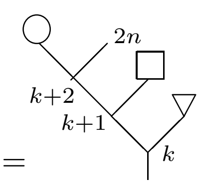
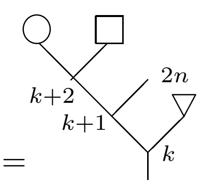
In our Tree Calculus we shall mostly compare the cardinalities of certain pairs of trunks, as shown in the following two examples.

Example 1. The two trunks

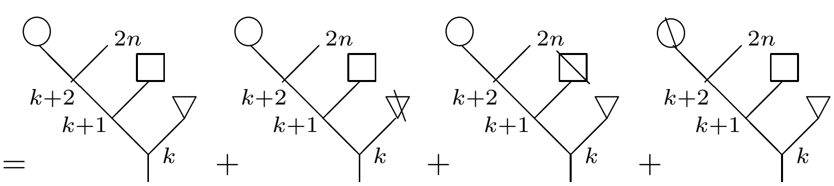
$[$  $, m]$ and $[$  $, m+2]$

have the same cardinalities, by using the bijection $\left(\begin{smallmatrix} m & m+1 & m+2 \\ m+2 & m & m+1 \end{smallmatrix}\right)$.

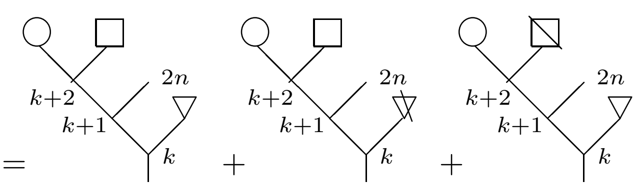
Example 2. To compare the cardinalities of the following two trunks

$C =$  and $D =$  ,

we decompose them according to the location of the one-child node:

$C =$ 

$:= C(\star) + C(\not\triangledown) + C(\boxbslash) + C(\oslash)$;

$D =$ 

$:= D(\star) + D(\not\triangledown) + D(\boxbslash)$.

In the above seven trunks of the two decompositions the symbols "$\square$," "$\triangledown$," "$\bigcirc$" without slash are not empty and do not contain the one-child node. Furthermore, note that $D = D(\star) + D(\not\triangledown) + D(\oslash)$ also holds.

Pivoting the two subtrees on each of the nodes k, $(k+1)$, $(k+2)$ in $C(\star)$ (resp. in $D(\star)$) yields the same trunk, as in (2.1). We then say that *subtree pivoting is permitted* on those nodes. Furthermore, the three subtrees "$\bigcirc$," "$\square$" and "$\bigtriangledown$" in $C(\star)$ play a symmetric role, while in $D(\star)$ only "$\bigcirc$" and "$\bigtriangledown$," on the one hand, and "$\square$" and "$\bigtriangledown$," on the other hand, have a symmetric role. Hence,

$$C(\star) = 2\,D(\star).$$

Now, remember that in each secant tree the one-child node lies at the rightmost position. This implies that *subtree pivoting on each of the ancestors of the one-child is not permitted.* Thus, subtree pivoting in $C(\bigtriangledown\!\!\!\!\!\star)$ and $D(\bigtriangledown\!\!\!\!\!\star)$ is only permitted on nodes $k+2$ and $k+1$. On the other hand, the two subtrees "$\bigcirc$" and "'$\square$" play a symmetric role in $C(\bigtriangledown\!\!\!\!\!\star)$, but not in $D(\bigtriangledown\!\!\!\!\!\star)$. Hence,

$$C(\bigtriangledown\!\!\!\!\!\star) = 2\,D(\bigtriangledown\!\!\!\!\!\star).$$

In $C(\boxbslash)$ and $D(\boxbslash)$ the two subtrees "$\bigcirc$" and "$\bigtriangledown$" play a symmetric role. Moreover, subtree pivoting on nodes $k+2$ is permitted in $C(\boxbslash)$, but not in $D(\boxbslash)$. Hence,

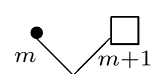
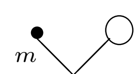
$$C(\boxbslash) = 2\,D(\boxbslash),$$

so that

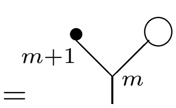
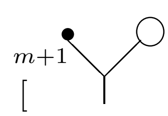
$$C - 2\,D = C(\obslash).$$

3. Proof that $(R\,1)$ holds

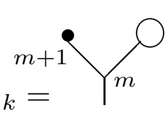
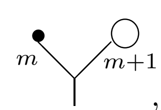
The decomposition

$\mathfrak{T}_{2n,m,k} =$  $+ \quad [$  $, \quad m+1]$

means that in each tree from $\mathfrak{T}_{2n,m,k}$ the node $(m+1)$ is, or is not, the sibling of the leaf m. In the next decomposition the node m is, or is not, the parent of the leaf $(m+1)$:

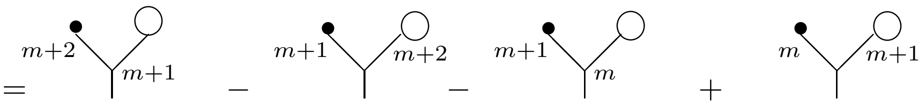
$\mathfrak{T}_{2n,m+1,k} =$  $+ \quad [$  $, m].$

As already explained in Example 1, Section 2, we can write:

$\underset{m}{\Delta}\mathfrak{T}_{2n,m,k} = \mathfrak{T}_{2n,m+1,k} - \mathfrak{T}_{2n,m,k} =$  $-$  ,

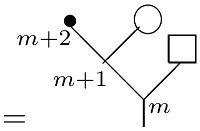
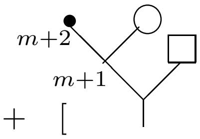
so that

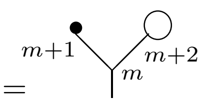
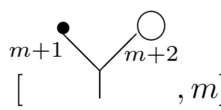
$$\underset{m}{\Delta^2}\,\mathfrak{T}_{2n,m,k} = (\mathfrak{T}_{2n,m+2,k} - \mathfrak{T}_{2n,m+1,k}) - (\mathfrak{T}_{2n,m+1,k} - \mathfrak{T}_{2n,m,k})$$

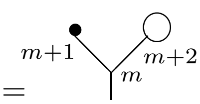
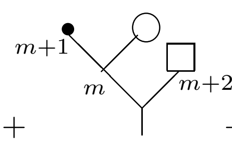
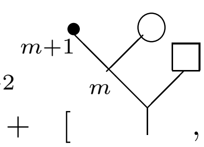
= 

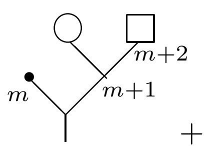
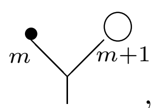
$:= A - B - C + D$.

Depending on the mutual positions of nodes m, $(m+1)$ and $(m+2)$ the further decompositions prevail, as again k still remains attached to $(2n)$:

$A =$  $+$ [ $, m] := A_1 + A_2$;

$B =$  $+$ [ $, m] := B_1 + B_2$;

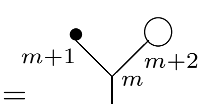
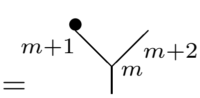
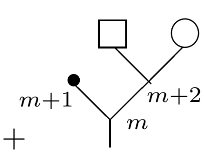
$C =$  $+$  $+$ [ $, m+2] := C_1 + C_2 + C_3$;

$D =$  $+$ [ $, m+2] := D_1 + D_2$.

In the above decompositions the subsets B_1 and C_1 are identical. Furthermore, $A_2 = C_3$ and $B_2 = D_2$. Accordingly, $\underset{m}{\Delta^2}\,\mathfrak{T}_{2n+1,m,k} = A_1 - 2\,B_1 - C_2 + D_1$. A further decomposition of those four terms, depending upon the occurrence of the one-child node, is to be worked out:

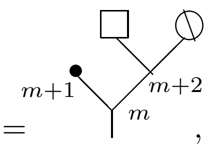
$$\begin{aligned}
A_1 &:= A_1(\star) + A_1(\boxbslash) + A_1(\oslash);\\
B_1 &:= B_1(\star) + B_1(\oslash);\\
C_2 &:= C_2(\star) + C_2(\boxbslash) + C_2(\oslash);\\
D_1 &:= D_1(\star) + D_1(\boxbslash) + D_1(\oslash).
\end{aligned}$$

Now, $B_1(\star)$ can be decomposed into

$B_1(\star) =$  $=$  $+$ 

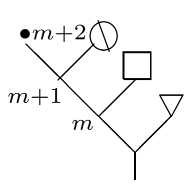
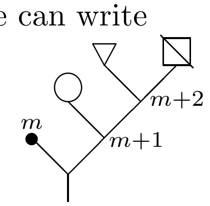
$= B_{1,1}(\star) + B_{1,2}(\star)$,

where "$\square$" and "$\bigcirc$" are supposed to be *non-empty* in $B_{1,2}(\star)$. By the Tree Calculus techniques developed in Section 2, we have $A_1(\star) = 2\,B_{1,2}(\star)$. On the other hand, $B_1(\oslash)$ can also be written

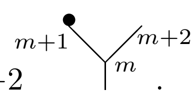
$B_1(\oslash) =$  ,

so that $A_1(\boxbslash) = 2\,B_1(\oslash)$ and $A_1(\oslash) = B_1(\oslash)$. Furthermore, $C_2(\star) = D_1(\star)$, $C_2(\boxbslash) = 2\,D_1(\boxbslash)$ and $C_2(\oslash) = D_1(\oslash)$. Altogether, $A_1 - 2\,B_1 - C_2 + D_1 = \big(A_1(\star) + A_1(\boxbslash) + A_1(\oslash)\big) - 2\big(B_{1,1}(\star) + B_{1,2}(\star) + B_1(\oslash)\big) - \big(C_2(\star) + C_2(\boxbslash) + C_2(\oslash)\big) + \big(D_1(\star) + D_1(\boxbslash) + D_1(\oslash)\big) = A_1(\oslash) - 2\,B_{1,1}(\star) - D_1(\boxbslash)$.

As m is supposed to be at least equal to 2, we can write

$A_1(\oslash) =$  , $D_1(\boxbslash) =$  ,

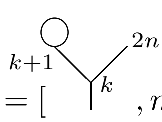
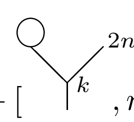
which shows that those two families are equal. Thus,

$$\Delta_m^2\, \mathfrak{T}_{2n,m,k} = -2\,B_{1,1}(\star) = -2$$  .

This expression is also equal to $-4\,\mathfrak{T}_{2n-2,m,k-2}$, because in each tree t from B_1 the nodes $(m+1)$ and $(m+2)$ are both leaves. Remove them, as well as the two edges going out of m, and subtract 2 from all the remaining nodes greater than $(m+2)$. The tree thereby derived belongs to $\mathfrak{T}_{2n-2,m,k-2}$. $\square$

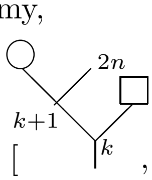
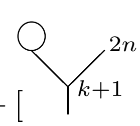
4. Tree Calculus for proving that $(R\,2)$ holds

With $n \geq 3$ and $2 \leq m \leq k-1 < k \leq 2n-3$ we have:

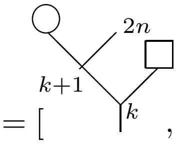
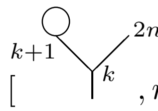
$\mathfrak{T}_{2n,m,k} = [$  $, m] + [$  $, m, k+1]$,

meaning that each tree from $\mathfrak{T}_{2n,m,k}$ has one of the two forms: either $k+1$ is incident to k, or not, and the leaf m is the end of the minimal chain.

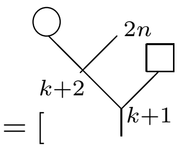
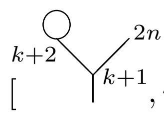
Using the same dichotomy,

$\mathfrak{T}_{2n,m,k+1} = [$  $, m] + [$  $, m, k]$.

As the second terms of the above two equations are in one-to-one correspondence by the transposition $(k, k+1)$ we have:

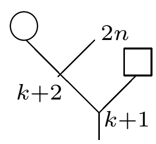
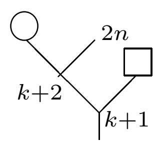
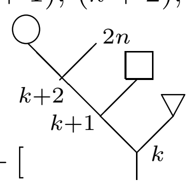
$\mathfrak{T}_{2n,m,k+1} - \mathfrak{T}_{2n,m,k} = [$  $, m] - [$  $, m] := B - A.$

In the same manner,

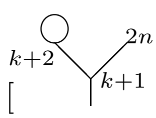
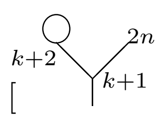
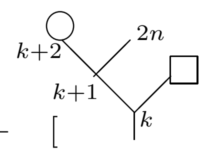
$\mathfrak{T}_{2n,m,k+2} - \mathfrak{T}_{2n,m,k+1} = [$  $, m] - [$  $, m] := D - C.$

Thus,
$$\begin{aligned}\Delta_k^2\,\mathfrak{T}_{2n,m,k} &= \big(\mathfrak{T}_{2n,m,k+2} - \mathfrak{T}_{2n,m,k+1}\big) - \big(\mathfrak{T}_{2n,m,k+1} - \mathfrak{T}_{2n,m,k}\big)\\ &= D - C - B + A.\end{aligned}$$

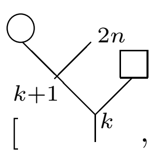
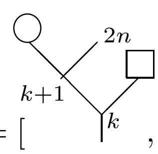
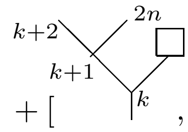
The further decompositions of the components of the previous sum depend on the mutual positions of the nodes k, $(k+1)$, $(k+2)$;

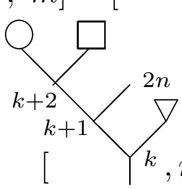
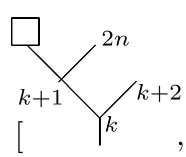
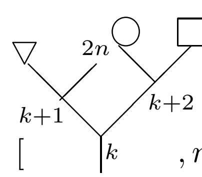
$D = [$  $, m] = [$  $, m, k\,] + [$  $, m]$

$:= D_1 + D_2;$

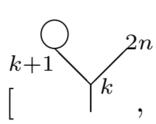
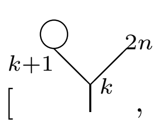
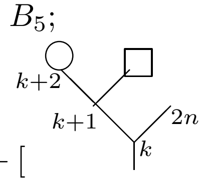
$C = [$  $, m] = [$  $, m, k\,] + [$  $, m]$

$:= C_1 + C_2;$

$B = [$  $, m] = [$  $, m, k+2] + [$  $, m]$

$+ [$  $, m] + [$  $, m] + [$  $, m]$

$:= B_1 + B_2 + B_3 + B_4 + B_5;$

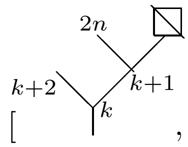
$A = [$  $, m] = [$  $, m, k+2\,] + [$  $, m]$

$:= A_1 + A_2,$

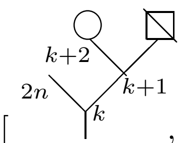
where "" and "$\square$" cannot be both empty in B_3 and B_5.

Obviously, $D_1 = B_1$ and $C_1 = A_1$, so that the sum $D - C - B + A$ may be written $(B_2 - B_4) - (C_2 - A_2) + (D_2 - B_3 - B_5) - 2\,B_2$, showing that the components fall in three categories: (1) B_2 and B_4 having one subtree to determine; (2) C_2 and A_2 having two such subtrees; (3) D_2, B_3 and B_5 having three of them.

(1) With the same notation as in (2.2) write: $B_2 = B_2(\star) + B_2(\boxbslash)$ and $B_4 = B_4(\star) + B_4(\boxbslash)$. First, $B_2(\star) = B_4(\star)$, but $B_2(\boxbslash) = 2\,B_4(\boxbslash)$. Hence,

$B_2 - B_4 = B_4(\boxbslash) = [$  $, m].$

(2) Next, $C_2 = C_2(\star) + C_2(\boxbslash) + C_2(\obslash)$ and $A_2 = A_2(\star) + A_2(\boxbslash) + A_2(\obslash)$. As before, $C_2(\star) = A_2(\star)$ and also $C_2(\obslash) = A_2(\obslash)$. However, $C_2(\boxbslash) = 2\,A_2(\boxbslash)$. Altogether,

$C_2 - A_2 = A_2(\boxbslash) = [$  $, m].$

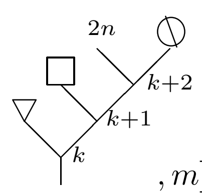
(3) The calculation of the third sum requires the following decompositions:

$$\begin{aligned}
D_2 &:= D_2(\star) + D_2(\not\bigtriangledown) + D_2(\boxbslash) + D_2(\obslash);\\
B_3 &:= B_3(\star) + B_3(\not\bigtriangledown) + B_3(\boxbslash);\\
B_5 &:= B_5(\star) + B_5(\not\bigtriangledown) + B_5(\boxbslash);
\end{aligned}$$

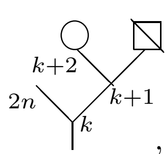
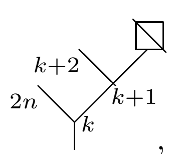
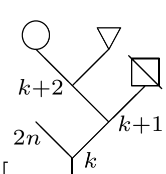
By the Tree Calculus techniques developed in Section 2, especially Example 2, we have

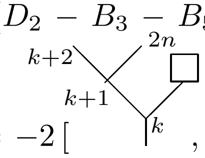
$$\begin{aligned}
D_2(\star) &= 2\,B_3(\star) = 2\,B_5(\star);\\
D_2(\not\bigtriangledown) &= 2\,B_3(\not\bigtriangledown) = 2\,B_3(\not\bigtriangledown);\\
D_2(\boxbslash) &= 2\,B_3(\boxbslash) = 2\,B_3(\boxbslash).
\end{aligned}$$

Hence,

$D_2 - B_3 - B_5 = D_2(\obslash) = [$  $, m].$

Now

$A_2(\boxbslash) = [$  $, m] = [$  $, m] + [$  $, m],$

where the two subtrees "$\bigcirc$" and "$\bigtriangledown$" are non-empty. The last identity can be rewritten: $A_2(\boxbslash) = B_4(\boxbslash) + D_2(\obslash)$. Hence, $\underset{k}{\Delta^2}\, \mathfrak{T}_{2n,m,k} = D - C - B + A = (B_2 - B_4) - (C_2 - A_2) + (D_2 - B_3 - B_5) - 2\,B_2 = B_4(\boxbslash) - A_2(\boxbslash) + D_2(\obslash) - 2\,B_2 = -2\,B_2 = -2\,[$ 

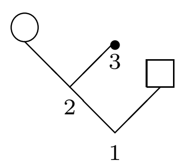
$, m]$. The last term is also equal to $-4\,\mathfrak{T}_{2n-2,m,k}$, because in each tree t from B_2 the nodes $(k+2)$ and $(2n)$ are both leaves. Remove them, as well as the two edges going out of $(k+1)$, change $(k+1)$ into $(2n-2)$ and subtract 2 from all the remaining nodes greater than $(k+2)$. The tree thereby derived belongs to $\mathfrak{T}_{2n-2,m,k}$. $\square$

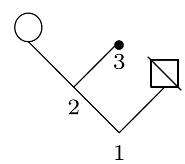
5. Proofs of Theorem 1.2, Proposition 1.3 and Theorem 1.4

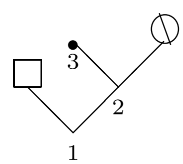
Proof of Theorem 1.2.

First top row. The trees t from $\mathfrak{T}_{2n}$ such that $\mathrm{eoc}(t) = 2$ contain the edge $2 \to 1$. Remove it and change each remaining node label j by $j-2$. We get a tree t' from $\mathfrak{T}_{2n-2}$ such that $\mathrm{pom}(t') = k-2$. The second equality is a consequence of (1.4). $\square$

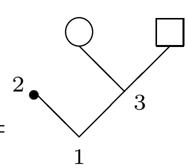
Second top row. As $k \ge 4$, the subtrees "$\square$" and "$\bigcirc$" below are non-empty, so that

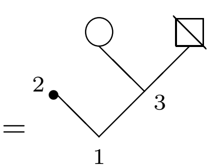
$f_{2n}(3,k) =$ 

$=$ 

$+$ 

$:= \quad A \quad = \quad A(\boxbslash) \quad + \quad A(\obslash);$

$f_{2n}(2,k) =$ 

$:= B =$ 

$= \quad B(\boxbslash),$

since $B(\star)$ is empty. Therefore, $A(\boxbslash) = 2\,B(\boxbslash)$, $A(\obslash) = B(\boxbslash)$ and then $f_{2,n}(3,k) = 3\,f_{2n}(2,k)$. $\square$

Rightmost column. The trees t from $\mathfrak{T}_{2n}$ such that $\mathrm{eoc}(t) = m$ and $\mathrm{pom}(t) = 2n-1$ contain the *rightmost* path $2n \to (2n-1) \to$. Remove it. What is left is a tree t' from $\mathfrak{T}_{2n-2}$ such that $\mathrm{eoc}(t') = m$. The second equality is a consequence of (1.4). $\square$

Next to rightmost column. If $t \in \mathfrak{T}_{2n}$ and $\mathrm{pom}(t) = 2n-1$, then t contains the rightmost path $2n \to (2n-1) \to$, as already noted. On the other hand, the node with label $(2n-2)$ is necessarily a leaf. The transposition

$(2n-2, 2n-1)$ transforms t into a tree t_1 such that $\mathrm{pom}(t_1) = 2n-2$. Also, removing the path $2n \to (2n-1)$ and rooting either the subtree ${}^{2n-1}\!\nwarrow\!\!\vee\!\!\nearrow^{2n}$, or the subtree ${}^{2n}\!\nwarrow\!\!\vee\!\!\nearrow^{2n-1}$, onto the leaf labeled $(2n-2)$ gives rise to two trees t_2, t_3 such that $\mathrm{pom}(t_2) = \mathrm{pom}(t_3) = 2n-2$. $\square$

Proof of Proposition 1.3. For (1.8) and (1.9) we only have to reproduce the proofs made in Theorem 1.2 for the first and second top rows, the parent of the maximum node playing no role.

To obtain $(R\,4)$ simply write $(R\,2)$ for $m = 2$ and rewrite it using the first identity of Theorem 1.2 dealing with the "first top row." Next, $(R\,3)$ is deduced from $(R\,4)$ by using identity (1.4). $\square$

Proof of Theorem 1.4. By induction, the row and column sums $f_{2n}(m, \bullet)$ and $f_{2n}(\bullet, k)$ of the matrices M_{2n} can be calculated by means of relations (1.8), (1.9) and, either $(R\,3)$, or $(R\,4)$. The first and second top rows (resp. rightmost and next to rightmost columns) of the matrix M_{2n} are known by Theorem 1.2. It then suffices to apply, either rule $(R\,1)$, from top to bottom, or rule $(R\,2)$ from right to left to obtain the remaining entries of the upper triangles of M_{2n}. $\square$

6. The lower triangles

Observe that for $2n \geq 6$ the non-zero entries of the first row, first column and last column of M_{2n} are identical and they differ from the entries in the bottom row. For instance, the sequence $5, 15, 21, 15, 5$ in M_8 appears three times, but the bottom row reads: $16, 16, 14, 10, 5$.

By Theorem 1.2 we already know that $f_{2n}(2,k) = f_{2n-2}(k-1, \bullet) = f_{2n}(k-1, 2n-1)$ $(3 \leq k \leq 2n-1)$. We also have: $f_{2n}(m, 1) = f_{2n-2}(m-1, \bullet)$ $(3 \leq m \leq 2n-1)$, by using this argument: each tree t from $\mathfrak{T}_{2n}$ satisfying $\mathrm{pom}(t) = 1$ has its leaf node $2n$ incident to root 1. Just remove the edge $2n \to 1$. Change each remaining label j to $j-1$. We get a secant tree t' belonging to $\mathfrak{T}_{2n-2}$. The mapping $t \mapsto t'$ is bijective; moreover, $\mathrm{eoc}(t') = \mathrm{eoc}(t) - 1$. As a summary,

$$f_{2n}(2,k) = f_{2n}(k-1, 2n-1) = f_{2n}(k, 1) \quad (3 \leq k \leq 2n-1). \tag{6.1}$$

Introduce a further statistic "ent" (short hand for "Entringer") on each tree t from $\mathfrak{T}_n$ (even for *tangent* trees) as follows: $\mathrm{ent}(t)$ is the label of the *rightmost* node of t. The distribution of "ent" is well-known, mostly associated with the model of the alternating permutations and traditionally called *Entringer* distribution (see, e.g., [En63], [Po82], [Po87], [GHZ10]). We also know how to calculate the generating function of that distribution and build up the *Entringer triangle* $(\mathrm{Ent}_n(j))$ $(1 \leq j \leq n-1;\ n \geq 2)$, as done in Table 6.1.

For instance, the leftmost 61 on the row $n = 7$ is the sum of all the entries in the previous row (including an entry 0 on the right!); the second 61 is the sum of the leftmost five entries; 56, the sum of the leftmost four entries; 46, the sum of the leftmost three entries; 32, the sum of the leftmost two entries and 16 is equal to the leftmost entry. Let $\mathrm{Ent}_n(j) = \#\{t \in \mathfrak{T}_n : \mathrm{ent}(t) = j\}$.

$j =$	1	2	3	4	5	6
$n =$ 2	1					
3	1	1				
4	2	2	1			
5	5	5	4	2		
6	16	16	14	10	5	
7	61	61	56	46	32	16

Table 6.1. The Entringer distribution

Proposition 6.1. *For $2 \le k \le 2n-2$ we have:*

$$f_{2n}(2n, k) = \mathrm{Ent}_{2n-2}(k-1). \tag{6.2}$$

Proof. Note that in a tree $t \in \mathfrak{T}_{2n}$ such that $\mathrm{eoc}(t) = 2n$ and $\mathrm{pom}(t) = k$ the leaf labeled $2n$ is the unique son of the node labeled k, which is also the rightmost node. Let $(1 = a_1) \to a_2 \to a_3 \to \cdots \to (a_{j-1} = k) \to (a_j = 2n)$ be the minimal chain of t. Form a new tree t' by means of the following changes:

(i) delete the path $\to a_{j-1} \to 2n$;

(ii) for $i = 1, 2, \ldots, j-2$ replace each node label a_i of the minimal chain by $a_{i+1} - 1$;

(iii) replace each other node label b by $b - 1$.

The label of the rightmost node of t' is then equal to the $\mathrm{pom}(t) - 1 = k - 1$. The mapping $t \mapsto t'$ is obvious bijective. ∎

Finally, the upper diagonal $\{f_{2n}(k+1, k) : (1 \le k \le 2n-1)\}$ of the lower triangle in each matrix M_{2n} can also be fully evaluated. First, in an obvious manner,

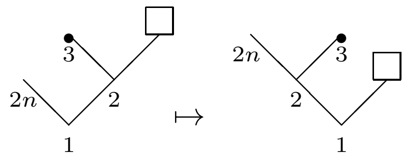
$$f_{2n}(2, 1) = f_{2n}(2n, 2n-1) = 0. \tag{6.3}$$

Second, the identities

$$f_{2n}(3, 2) = 2\, f_{2n}(3, 1) = f_{2n}(2n-1, 2n-2) = 2\, f_{2n}(2n, 2n-2) \tag{6.4}$$

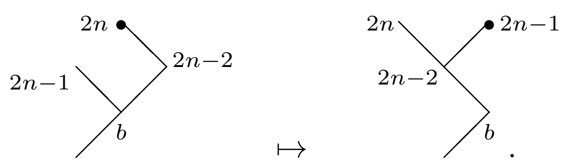
(also equal to $f_{2n-4}(\bullet, \bullet) = E_{2n-4}$) can be proved as follows. To each tree $t \in \mathfrak{T}_{2n}$ such that $\mathrm{eoc}(t) = 3$, $\mathrm{pom}(t) = 1$ there correspond *two* trees,

whose “eoc” and “pom” are equal to 2 and 3, respectively, as illustrated by the diagram:

$f_{2n}(3,1) =$ 

$= f_{2n}(3,2).$

Likewise, each tree $t \in \mathfrak{T}_{2n}$ such that $\mathrm{eoc}(t) = 2n$ and $\mathrm{pom}(t) = 2n-2$ necessarily has its rightmost four nodes equal to $(2n-1), b, 2n, (2n-2)$ in that order, with b being a node less than $(2n-2)$. In particular, the latter node is its rightmost (one-child) node. To such a tree there correspond two trees, whose “eoc” and “pom” are equal to $(2n-1)$, $(2n-2)$, respectively, as illustrated by the next diagram. In particular, the node b becomes the rightmost node of those two such trees.



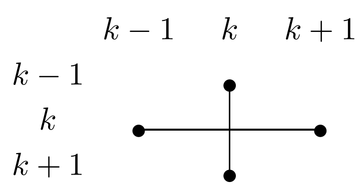
. ∎

Proposition 6.2. *We have the crossing equalities:*

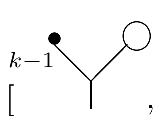
$$f_{2n}(k-1,k) + f_{2n}(k+1,k) = f_{2n}(k,k-1) + f_{2n}(k,k+1), \tag{6.5}$$

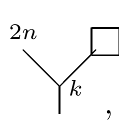
for $3 \le k \le 2n-2$.

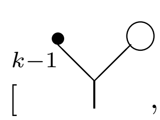
The involved entries are located on the four bullets drawn in the following diagramme.

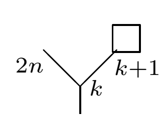


Proof. Let i, j be two different integers from the set $\{(k-1), k, (k+1)\}$. Say that i and j are *connected* in a tree t, if the tree contains the edge i—j, or if i and j are brothers and of them is the end of the minimal chain of t. Each of the four ingredients of the previous identity is now decomposed into five terms, depending on whether the nodes $(k-1)$, k, $(k+1)$ are connected or not, namely: no connectedness; only $k, (k+1)$ connected; $(k-1), k$ connected; $(k-1), (k+1)$ connected; all connected. Thus,

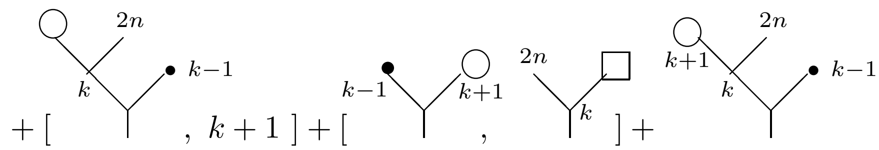
$f_{2n}(k-1,k) = [$ 

, 

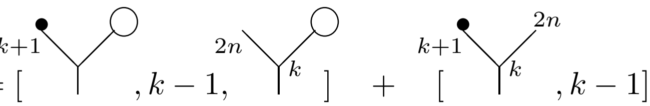
$, k+1] + [$ 

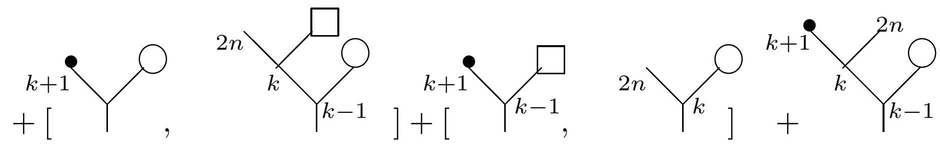
, 

$]$

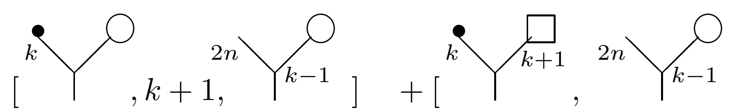


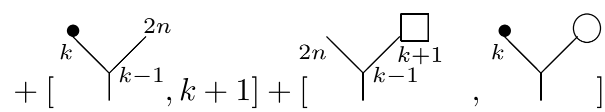
$:= A_1 + A_2 + A_3 + A_4 + A_5;$

$f_{2n}(k+1,k) =$ 

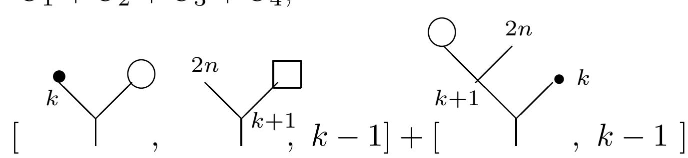


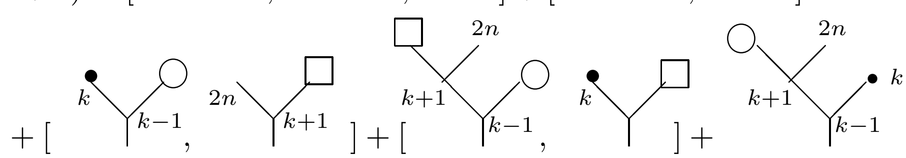
$:= B_1 + B_2 + B_3 + B_4 + B_5;$

$f_{2n}(k,k-1) =$ 



$:= C_1 + C_2 + C_3 + C_4;$

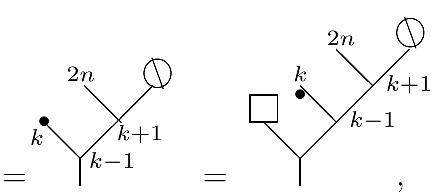
$f_{2n}(k,k+1) =$ 



$:= D_1 + D_2 + D_3 + D_4 + D_5.$

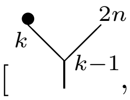
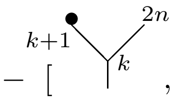
Now, the following identities hold: $A_1 = C_1$, $A_2 = C_4$, $A_3 = D_2$, $A_4 = C_2$, $B_1 = D_1$, $B_3 = D_4$, $B_4 = D_3$, so that $\sum_i (A_i + B_i) - \sum_i (C_i + D_i) = (B_5 - D_5) - (C_3 - A_5 - B_2)$.

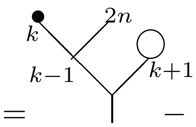
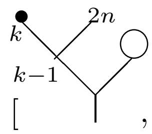
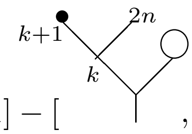
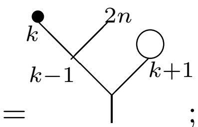
As before, we may write $B_5 = B_5(\star) + B_5(\oslash)$, $D_5 = D_5(\star) + D_5(\oslash)$. As $B_5(\star) = D_5(\star)$ and $B_5(\oslash) = 2\, D_5(\oslash)$, we get:

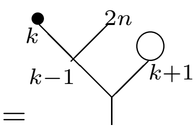
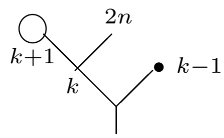
(6.6) $B_5 - D_5 = D_5(\oslash) =$  ,

as k is supposed to be greater than 3.

Next,

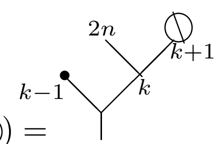
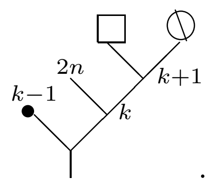
$C_3 - B_2 = [$  $, k+1] - [$  $, k-1]$

$=$  $- [$  $, k+1] - [$  $, k-1] =$  ;

$C_3 - B_2 - A_5 =$  $-$ 

$:= E - F$

$= (E(\star) - F(\star)) + (E(\oslash) - F(\oslash))$

(6.7) $= F(\oslash) =$  $=$  .

By comparing the evaluations (6.6) and (6.7) we get: $(B_5 - D_5) - (C_3 - A_5 - B_2) = 0$. This completes the proof of (6.5). $\square$

The entries $f_{2n}(3,2)$ and $f_{2n}(2n-1, 2n-2)$, belonging to the upper diagonal of the lower triangle matrix M_{2n}, being evaluated by identity (6.4), and the entries of the upper triangle being known by Theorem 1.4, we can apply the crossing equalities (6.5), starting with $k = 3$, and obtain all the values of the entries of that upper diagonal. Altogether, besides the entries of the upper triangle, all the entries lying on the border of the lower triangle can be calculated, as illustrated in boldface in the following matrix M_8.

$M_8 =$

$k =$	1	2	3	4	5	6	7	$f_8(m, .)$
$m = 2$	.	.	**5**	**15**	**21**	**15**	**5**	**61**
3	**5**	**10**	.	**45**	**63**	**45**	**15**	**183**
4	**15**	35	**50**	.	**101**	**63**	**21**	**285**
5	**21**	54	86	**106**	.	**45**	**15**	**327**
6	**15**	46	82	87	**50**	.	**5**	**285**
7	**5**	22	46	60	40	**10**	.	**183**
8	.	**16**	**16**	**14**	**10**	**5**	.	**61**
$f_8(., k)$	**61**	**183**	**285**	**327**	**285**	**183**	**61**	$\mathbf{E_8 = 1385}$

Table 6.1: the matrix M_8, bold-faced entries analytically evaluated.

7. Generating functions for the $f_{2n}(m,k)$

The calculation of the generating functions for the $f_{2n}(m,k)$ is similar to the calculation made for the tangent tree case in a our previous paper [FH13]. Recall the definition and some basic properties of the Poupard matrix. Let $G=(g_{i,j})$ $(i\ge 0,\, j\ge 0)$ be an infinite matrix with nonnegative integral entries. Say that G is a *Poupard matrix*, if for every $i\ge 0$, $j\ge 0$ the following identity holds:

$$g_{i,j+2}-2\,g_{i+1,j+1}+g_{i+2,j}+4\,g_{i,j}=0. \tag{7.1}$$

Remark. Last coefficient is 4, not 2 as in [FH13].

Let $G(x,y):=\sum_{i\ge 0,\, j\ge 0} g_{i,j}\,(x^i/i!)\,(y^j/j!)$; $R_i(y):=\sum_{j\ge 0} g_{i,j}\,(y^j/j!)$ $(i\ge 0)$; $C_j(x):=\sum_{i\ge 0} g_{i,j}\,(x^i/i!)$ $(j\ge 0)$ be the exponential generating functions for the matrix itself, its rows and columns, respectively. Propositions 9.1 and 9.2 in [FH13] can be rewritten as follows.

Proposition 7.1. *The following four properties are equivalent.*
(i) $G=(g_{i,j})$ $(i\ge 0,\, j\ge 0)$ *is a Poupard matrix;*
(ii) $R_i''(y)-2\,R_{i+1}'(y)+R_{i+2}(y)+4\,R_i(y)=0$ *for all* $i\ge 0$*;*
(iii) $C_j''(x)-2\,C_{j+1}'(x)+C_{j+2}(x)+4\,C_j(x)$ *for all* $j\ge 0$*;*
(iv) $G(x,y)$ *satisfies the partial differential equation:*

$$\frac{\partial^2 G(x,y)}{\partial x^2}-2\,\frac{\partial^2 G(x,y)}{\partial x\,\partial y}+\frac{\partial^2 G(x,y)}{\partial y^2}+4\,G(x,y)=0. \tag{7.2}$$

Proposition 7.2. *Let $G(x,y)$ be the exponential generating function for a Poupard matrix G. Then,*

$$G(x,y)=A(x+y)\,\cos(2\,y)+B(x+y)\,\sin(2\,y), \tag{7.3}$$

where $A(y)$ and $B(y)$ are two arbitrary series.

The entries of the *upper triangles* of the matrices (M_{2n}) (see Table 1.1) are now recorded as entries of infinite matrices $(\Omega^{(p)})$ $(p\ge 1)$ by

$$\omega_{i,j}^{(p)}:=\begin{cases}0, & \text{if } i+j\equiv p \bmod 2;\\ f_{2n}(m,k), & \text{if } i+j\not\equiv p \bmod 2;\end{cases} \tag{7.4}$$

with $m:=p+1$, $k:=p+j+2$, $2n:=p+i+j+3$. Conversely, $i:=2n-k-1$, $j:=k-m-1$, $p:=m-1$. In particular, the first one $\Omega^{(1)}=(\omega_{i,j}^{(1)})$ $(i,j\ge 0)$ contains the first rows of the upper triangles,

displayed as counter-diagonals. Furthermore, a counter-diagonal with zero entries is placed between two successive rows.

$$\Omega^{(1)} =$$

$$\begin{array}{c|c}
 & \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{array}{c} 0\\1\\2\\3\\4\\5\\6\\7 \end{array} &
\begin{pmatrix}
f_4(2,3) & 0 & f_6(2,5) & 0 & f_8(2,7) & 0 & f_{10}(2,9) & 0 \\
0 & f_6(2,4) & 0 & f_8(2,6) & 0 & f_{10}(2,8) & 0 & \cdots \\
f_6(2,3) & 0 & f_8(2,5) & 0 & f_{10}(2,7) & 0 & \cdots & \\
0 & f_8(2,4) & 0 & f_{10}(2,6) & 0 & \cdots & & \\
f_8(2,3) & 0 & f_{10}(2,5) & 0 & \cdots & & & \\
0 & f_{10}(2,4) & 0 & \cdots & & & & \\
f_{10}(2,3) & 0 & \cdots & & & & & \\
0 & \cdots & & & & & &
\end{pmatrix}
\end{array}$$

$$= \begin{array}{c|c}
 & \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{array}{c} 0\\1\\2\\3\\4\\5\\6\\7 \end{array} &
\begin{pmatrix}
1 & 0 & 1 & 0 & 5 & 0 & 61 & 0 \\
0 & 3 & 0 & 15 & 0 & 183 & 0 & \cdots \\
1 & 0 & 21 & 0 & 285 & 0 & \cdots & \\
0 & 15 & 0 & 327 & 0 & \cdots & & \\
5 & 0 & 285 & 0 & \cdots & & & \\
0 & 183 & 0 & \cdots & & & & \\
61 & 0 & \cdots & & & & & \\
0 & \cdots & & & & & &
\end{pmatrix}
\end{array}.$$

Proposition 7.3. *Every matrix $\Omega^{(p)}$ $(p \geq 1)$ is a Poupard matrix.*

Proof. Using Definition (7.4) we have

$$\begin{aligned}
\omega_{i,j+2}^{(p)} - 2\,\omega_{i+1,j+1}^{(p)} + \omega_{i+2,j}^{(p)} + 4\,\omega_{i,j}^{(p)} \\
&= f_{2n+2}(m,k+2) - 2\,f_{2n+2}(m,k+1) \\
&\qquad + f_{2n+2}(m,k) + 4\,f_{2n}(m,k) \\
&= \mathop{\Delta}_{k} f_{2n+2}(m,k) + 4\,f_{2n}(m,k) = 0,
\end{aligned}$$

by rule $(R\,2)$. ∎

The row labeled i of $\Omega^{(p)}$ will be denoted by $\Omega_{i,\bullet}^{(p)}$ and the exponential generating function for that row by $\Omega_{i,\bullet}^{(p)}(y) = \sum_{j\geq 0} \omega_{i,j}^{(p)} y^j/j!$. Also, $\Omega^{(p)}(x,y) := \sum_{i\geq 0} \Omega_{i,\bullet}^{(p)}(y)x^i/i!$ will be the double exponential generating function for the matrix $\Omega^{(p)}$.

Proposition 7.4. *For all $p \geq 1$ we have:*

$$\Omega_{0,\bullet}^{(p+1)}(y) = \Omega_{p,\bullet}^{(1)}(y) \quad \textit{and} \quad \Omega_{1,\bullet}^{(p)}(y) = 3\,\frac{d}{dy}\Omega_{0,\bullet}^{(p)}(y) = 3\,\frac{d}{dy}\Omega_{p-1,\bullet}^{(1)}(y).$$

Proof. For the first identity it suffices to prove $\omega_{0,j}^{(p+1)} = \omega_{p,j}^{(1)}$, that is

$$f_{2n}(m, 2n-1) = f_{2n}(2, 2n+1-m).$$

This is true by the symmetry property of the Poupard triangle, as proved in Corollary 1.3 in [FH12]. For the second identity it suffices to prove $\omega_{1,j}^{(p)} = 3\omega_{0,j+1}^{(p)}$, that is

$$f_{2n}(m, 2n-2) = 3f_{2n}(m, 2n-1).$$

This is true by Theorem 1.2. $\square$

As x and y play a symmetric role in (7.3), the solution in (7.3) may also be written

$$G(x,y) = A(x+y)\,\cos(2\,x) + B(x+y)\,\sin(2\,x),$$

so that the generating function of each matrix $\Omega^{(p)}$ is of the form

$$\Omega^{(p)}(x,y) = A(x+y)\,\cos(2\,x) + B(x+y)\,\sin(2\,x).$$

Hence, $\Omega^{(p)}(x,y)\,\Big|_{\{x=0\}} = \Omega_{0,\bullet}^{(p)}(y) = A(y)$. Also,

$$\begin{aligned}(\partial/\partial x)\Omega^{(p)}(x,y) &= ((\partial/\partial x)A(x+y))\,\cos(2\,x) + A(x+y)(-2)\sin(2\,x)\\ &\quad + ((\partial/\partial x)B(x+y))\,\sin(2\,x) + B(x+y)\,2\,\cos(2\,x)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial x}\Omega^{(p)}(x,y)\,\Big|_{\{x=0\}} &= \frac{\partial}{\partial x}A(x+y)\,\Big|_{\{x=0\}} + 2\,B(x+y)\,\Big|_{\{x=0\}}\\ &= \frac{d}{dy}A(y) + 2\,B(y)\\ &= \Omega_{1,\bullet}^{(p)}(y).\end{aligned}$$

By Proposition 7.4 we have $A(y) = \Omega_{p-1,\bullet}^{(1)}(y)$ and $B(y) = (\Omega_{1,\bullet}^{(p)}(y) - \frac{d}{dy}A(y))/2 = \Big(3\frac{d}{dy}\Omega_{p-1,\bullet}^{(1)}(y) - \frac{d}{dy}\Omega_{p-1,\bullet}^{(1)}(y)\Big)/2 = \frac{d}{dy}\Omega_{p-1,\bullet}^{(1)}(y)$. Hence,

$$\Omega^{(p)}(x,y) = \cos(2\,x)\,\Omega_{p-1,\bullet}^{(1)}(x+y) + \sin(2\,x)\,\frac{\partial}{\partial y}\Omega_{p-1,\bullet}^{(1)}(x+y). \tag{7.5}$$

First, make the evaluation of $\Omega^{(1)}(x,y)$. The row labeled 0 of the matrix $\Omega^{(1)}$ reads: 1, 0, 1, 0, 5, 0, 61, ..., which is the sequence of the coefficients of the Taylor expansion of $\sec y$. Thus, $\Omega_{0,\bullet}^{(1)}(y) = \sec y$. Taking $p=1$ in (7.5) we get

$$\begin{aligned}\Omega^{(1)}(x,y) &= \sec(x+y)\cos(2\,x) + \sec(x+y)\,\tan(x+y)\,\sin(2\,x)\\ &= \frac{\cos(x-y)}{\cos^2(x+y)}.\end{aligned} \tag{7.6}$$

For further use let us also calculate the partial derivative of $\Omega^{(1)}(x,y)$ with respect of y:

$$\frac{\partial}{\partial y}\Omega^{(1)}(x,y) = \frac{1}{\cos^3(x+y)}\Big(\sin(x-y)\cos(x+y) + 2\cos(x-y)\sin(x+y)\Big)$$
$$= \frac{1}{2\,\cos^3(x+y)}\big(\sin(2y) + 3\,\sin(2x)\big). \tag{7.7}$$

Now, define

$$\Omega(x,y,z) := \sum_{p\geq 1}\Omega^{(p)}(x,y)\frac{z^{p-1}}{(p-1)!} \tag{7.8}$$

and make use of (7.5)—(7.8):

$$\begin{aligned}
\Omega(x,y,z) &= \cos(2x)\sum_{p\geq 1}\Omega^{(1)}_{p-1,\bullet}(x+y)\frac{z^{p-1}}{(p-1)!} \\
&\qquad + \sin(2x)\,\frac{\partial}{\partial y}\Big(\sum_{p\geq 1}\Omega^{(1)}_{p-1,\bullet}(x+y)\frac{z^{p-1}}{(p-1)!}\Big) \\
&= \cos(2x)\,\Omega^{(1)}(z,x+y) + \sin(2x)\,\frac{\partial}{\partial y}\,\Omega^{(1)}(z,x+y) \\
&= \cos(2x)\,\frac{\cos(x+y-z)}{\cos^2(x+y+z)} \\
&\qquad + \sin(2x)\,\frac{\sin(2(x+y)) + 3\,\sin(2z)}{2\,\cos^3(x+y+z)} \\
&= \frac{1}{2\,\cos^3(x+y+z)}\Big(\cos(2x)\,\big(\cos(2(x+y)) + \cos(2z)\big) \\
&\qquad + \sin(2x)\,\big(\sin(2(x+y)) + 3\sin(2z)\big)\Big) \\
&= \frac{1}{2\,\cos^3(x+y+z)}\Big(\cos(2y) + 2\,\cos(2(x-z)) - \cos(2(z+x))\Big).
\end{aligned} \tag{7.9}$$

By definition of the $\omega^{(p)}_{i,j}$'s we get

$$\Omega(x,y,z) = \sum_{p,i,j}\omega^{(p)}_{i,j}\,\frac{z^{p-1}}{(p-1)!}\,\frac{x^i}{i!}\,\frac{y^j}{j!} \quad (p\geq 1,\ i\geq 0,\ j\geq 0);$$
$$= \sum_{k,m,n} f_{2n}(m,k)\frac{x^{2n-k-1}}{(2n-k-1)!}\,\frac{y^{k-m-1}}{(k-m-1)!}\,\frac{z^{m-2}}{(m-2)!}, \tag{7.10}$$

the latter sum over the set $\{3\leq m+1\leq k\leq 2n-1\}$. We have proved Theorem 1.5.

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Dominique Foata
Institut Lothaire
1, rue Murner
F-67000 Strasbourg, France
`foata@unistra.fr`

Guo-Niu Han
I.R.M.A. UMR 7501
Université de Strasbourg et CNRS
7, rue René-Descartes
F-67084 Strasbourg, France
`guoniu.han@unistra.fr`